BLACK-AND-WHITE THRESHOLD GRAPHS

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Abstract

Let k be a natural number. We introduce k-*threshold graphs*. We show that there exists an $O(n^3)$ algorithm for the recognition of k-threshold graphs for each natural number k. k-Threshold graphs are characterized by a finite collection of forbidden induced subgraphs. For the case $k = 2$ we characterize the *partitioned 2-threshold graphs* by forbidden induced subgraphs. We introduce *restricted*, and *special 2-threshold graphs*. We characterize both classes by forbidden induced subgraphs. The restricted 2-threshold graphs coincide with the *switching class of threshold graphs*. This provides a decomposition theorem for the switching class of threshold graphs.

1 Introduction

A graph is a pair $G = (V, E)$ where V is a finite, nonempty set and where E is a set of two-element subsets of V. We call the elements of V the vertices or points of the graph. We denote the elements of E as (x, y) where x and y are vertices. We call the elements of E the edges of the graph. For two sets A and B, we use $A + B$ and $A − B$ to denote $A \cup B$ and $A \setminus B$ respectively. For a set A and an element x, denote $A \cup \{x\}$ and $A \setminus \{x\}$ as $A + x$ and $A − x$ respectively. If $e = (x, y)$ is an edge of a graph then we call x and y the endpoints of e and we say that x and y are adjacent. The *open neighborhood* of a vertex x is the set of vertices y such that $(x, y) \in E$. We denote open neighborhood by $N(x)$. Define the *closed neighborhood* of x by $N(x) + x$. We use $N[x]$ to denote the closed neighborhood of x. Define $\Delta(x) = V − N[x]$. We call $\Delta(x)$ the *anti-neighborhood* of x. The degree of a vertex x is the cardinality of $N(x)$. Let $W \subseteq V$ and let $W \neq \varnothing$. The graph $G[W]$ induced by W has W as its set of vertices and it has those edges of E that have both endpoints in W. If $W \subset V$, $W \neq V$, then we write $G − W$ for the graph induced by $V \setminus W$. In case W consists of a single vertex x we write $G − x$ instead of $G − \{x\}$. We usually denote the number of vertices of a graph by n.

A path is a graph of which the vertices can be linearly ordered such that the pairs of consecutive ver-
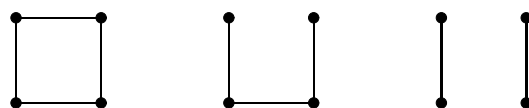
Figure 1: C_4, P_4 and $2K_2$

tices form the set of edges of the graph. We call the first and last vertex in this ordering the terminal vertices of the path. We denote a path with n vertices by P_n. To denote a specific ordering of the vertices we use the notation $[x_1, \ldots, x_n]$. Let G be a graph. Two vertices x and y of G are connected by a path if G has an induced subgraph which is a path with x and y as terminals. Being connected by a path is an equivalence relation on the set of vertices of the graph. The equivalence classes are called the components of G. A cycle consists of a path with at least three vertices with one additional edge that connects the two terminals of the path. We denote a cycle with n vertices by C_n. The length of a path or a cycle is its number of edges.

2 k-Threshold graphs

Threshold graphs were introduced in (Chvátal et al. 1973) using a concept called 'threshold dimension.' There is a lot of information about threshold graphs in the book (Mahadev & Peled 1995), and there are chapters on threshold graphs in the book (Golumbic 2004) and in the survey (Brandstädt et al. 1999).

There are many ways to define threshold graphs. We choose the following way (Chvátal et al. 1973, Brandstädt et al. 1999). An isolated vertex in a graph G is a vertex without neighbors. A universal vertex is a vertex that is adjacent to all other vertices. A pendant vertex is a vertex with exactly one neighbor.

Definition 1. *A graph* $G = (V, E)$ *is a* threshold graph *if every induced subgraph has an isolated vertex or a universal vertex.*

A graph G is a threshold graph if and only if G has no induced P_4, C_4, nor $2K_2$ (Chvátal et al. 1973).

Figure 2: A house, a C_5, a gem, a butterfly, and a bull

A decomposition tree for a graph $G = (V, E)$ is a pair (T, f) where T is a rooted binary tree and where f is a bijection from the vertices of G to the leaves of T.

Let G be a threshold graph. We can construct a decomposition tree for G as follows. For each internal

node, including the root, the right subtree consists of a single leaf. Each internal node is labeled with a $\oplus$-operator or a $\otimes$-operator. Consider an internal node and let W be the set of vertices that are mapped to leaves in the left subtree. The $\oplus$-operator adds the vertex that is mapped to the leaf in the right subtree as an isolated vertex to the graph induced by W. The $\otimes$-operator adds the vertex that is mapped to the leaf in the right subtree as a universal vertex to the graph induced by W.

Let $C = \{1, \ldots, k\}$ be a set of k colors. A coloring of a graph $G = (V, E)$ is a map $c : V \to C$.

Definition 2. *A graph* $G = (V, E)$ *is a* k-threshold graph *if there is a coloring of* G *with* k *colors such that for every nonempty set* $W \subseteq V$ *the graph* $G[W]$ *has a vertex which is either isolated, or a vertex* x *which is adjacent to exactly all vertices of color* i *in* $W - x$ *for some color* i.

A k-threshold graph G has a decomposition tree (T, f) where T is a rooted binary tree and f is a bijection from the leaves of T to the vertices of G. Each leaf has one color chosen from a set of k colors. For each internal node the right subtree consists of a single leaf. Each internal node including the root is labeled with a $\oplus$-operator or a $\otimes_i$-operator. The $\oplus$-operator adds the vertex that is mapped to the leaf in the right subtree as an isolated vertex. The $\otimes_i$-operator makes the vertex of the right leaf adjacent exactly to the vertices of color i in the left subtree.

Consider the class of k-threshold graphs. Notice that the class is hereditary; that is, if G is a k-threshold graph then so is every induced subgraph of G. Any hereditary class of graphs is characterized by a set of forbidden induced subgraphs. These are the minimal elements in the induced subgraph relation that are not in the class. For example, a graph is a threshold graph if and only if it has no induced $2K_2$, C_4 or P_4. In the following theorem we show that this characterization is finite for k-threshold graphs for any fixed number k.

Theorem 1. *Let* k *be a natural number. The class of* k-*threshold graphs is characterized by a finite collection of forbidden induced subgraphs.*

Proof. To prove this theorem we use the technique introduced by Pouzet (Pouzet 1985).

Let $T_1, T_2, \ldots$ be a collection of rooted binary trees with points labeled from some finite set. We write $T_i \prec T_j$ if there exists an injective map h from the points of T_i to the points of T_j such that

1. the label of a point a in T_i is equal to the label of the point $h(a)$ in T_j, and

2. for every pair of points a and b in T_i, their lowest common ancestor is mapped to the lowest common ancestor of $h(a)$ and $h(b)$ in T_j, and

3. if a and b are points of T_i with lowest common ancestor c such that a is in the left subtree of c and b is in the right subtree of c then $h(a)$ is in the left subtree of $h(c)$ and $h(b)$ is in the right subtree of $h(c)$ in T_j.

Let $T_1, T_2, \ldots$ be an infinite sequence of rooted binary trees with points labeled from some finite set. Kruskal's theorem (Kruskal 1960) states that there exist integers $i < j$ such that $T_i \prec T_j$.

Assume that the class of k-threshold graphs has an infinite collection of minimal forbidden induced subgraphs, say $G_1, G_2, \ldots$. In each G_i single out one

vertex r_i and let $G'_i = G_i - r_i$. Then G'_i is a k-threshold graph. For each i consider a decomposition tree (T_i, f_i) for G'_i as described above. We add one more label to each leaf of T_i. This label is 1 if the vertex that is mapped to that leaf is adjacent to r_i and it is 0 otherwise.

When we apply Kruskal's theorem to the labeled binary trees T_i that represent the graphs G'_i we may conclude that there exist $i < j$ such that G'_i is an induced subgraph of G'_j. By virtue of the extra label we have also that G_i is an induced subgraph of G_j. This is a contradiction because we assume that the graphs G_i are minimal forbidden induced subgraphs. This proves the theorem. $\square$

Higman's lemma (Higman 1952) preludes Kruskal's theorem. It deals with finite sequences over a finite alphabet instead of trees. Instead of Kruskal's theorem we could have used Higman's lemma to prove Theorem 1.

Let $\mathcal{O}_k$ be the set of forbidden induced subgraphs for k-threshold graphs. The set $\mathcal{O}_k$ is called the obstruction set for k-threshold graphs.

The most natural way to express and classify graph-theoretic problems is by means of logic. In monadic second order logic a (finite) sentence is a formula that uses quantifiers $\forall$ and $\exists$. The quantification is over vertices, edges, and subsets of vertices and edges. Relational symbols are $\neg$, $\in$, $=$, and, or, $\subseteq$, $\cup$, $\cap$, and the logical implication $\Rightarrow$. Some of these are superfluous. Although the minimization or maximization of the cardinality of a subset is not part of the logic, one usually includes them.

A restricted form of this logic is where one does not allow quantification over subsets of edges. The C_2MS-logic is such a restricted monadic second-order logic where one can furthermore use a test whether the cardinality of a subset is even or odd.

Courcelle proved that problems that can be expressed in C_2MS-logic can be solved in $O(n^3)$ time for graphs of bounded rankwidth (see (Courcelle & Oum 2007, Hliněný et al. 2008)). Consider a decomposition tree (T, f) for a graph $G = (V, E)$. Let ℓ be a line in this tree. Let V_ℓ be the set of vertices that are mapped to the leaves in the subtree rooted at ℓ. The cutmatrix of ℓ is the submatrix of the 0/1-adjacency matrix of G that has rows indexed by the vertices of V_ℓ and that has columns indexed by the vertices of $V - V_\ell$. A graph has rankwidth k if every cutmatrix has rank over GF[2] at most k. For each natural number k there exists an $O(n^3)$ algorithm that constructs a decomposition tree for a graph G of rankwidth k if it exists (Hliněný & Oum 2008).

Let (T, f) be a decomposition tree for a k-threshold graph. Then every cutmatrix is of the following shape:

$$\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

where I is a submatrix of the $k \times k$ identity matrix. By 'having this shape' we mean that we left out multiple copies of the same row or column. It follows that k-threshold graphs have rankwidth k.

Theorem 2. *Let* k *be a natural number. There exists an* $O(n^3)$ *algorithm which checks whether a graph* G *with* n *vertices is a* k-*threshold graph.*

Proof. k-Threshold graphs have rankwidth k. C_2MS-Problems can be solved in $O(n^3)$ time for graphs of bounded rankwidth (Courcelle & Oum 2007, Hliněný & Oum 2008, Oum 2005). The definition of k-threshold graphs can be formulated in this logic.

An other proof goes as follows. Let $\mathcal{O}_k$ be the obstruction set for k-threshold graphs. We can formulate the existence of a graph in $\mathcal{O}_k$ as an induced subgraph in C_2MS-logic. Note that this proof is non-constructive; Kruskal's theorem does not provide the forbidden induced subgraphs. $\qquad\square$

3 Special 2-threshold graphs

There are many drawbacks to the solution given in the previous section. The algorithm that constructs a decomposition tree of bounded rankwidth is far from easy. By the way, it is described in terms of matroids instead of graphs. Furthermore, the constants that are involved quickly grow out of space when k increases.

Notice that threshold graphs can be recognized by sorting the vertices according to increasing degrees. Either the vertex with highest degree is a universal vertex or the vertex with lowest degree is an isolated vertex. Delete such a vertex which is either isolated or universal and repeat the process. The graph is a threshold graph if and only if this process ends with a single vertex.

In the remainder of this paper we restrict our attention to the case where $k = 2$. We consider colorings of vertices with two colors, say black and white. If G is a 2-threshold graph with a black-and-white coloring of the vertices then the black vertices induce a threshold graph and the white vertices induce a threshold graph.

Definition 3. *A 2-threshold graph is* special *if it has a black-and-white coloring and a decomposition tree with only* $\oplus$- *and* $\otimes_w$-*operators.*

A clique in a graph is a nonempty subset of vertices such that every pair in it is adjacent. An independent set in a graph is a nonempty subset of vertices with no edges between them.

A graph is bipartite if there is a partition of the vertices into two independent sets. One part of the partition may be empty. A bipartite graph is complete bipartite if any two vertices in different parts of the partition are adjacent.

A threshold order of a graph is a linear ordering $[v_1, \ldots, v_n]$ of the vertices such that for every pair v_i and v_j with $i < j$

$$N(v_i) \subseteq N(v_j) \quad \text{if } v_i \text{ and } v_j \text{ are nonadjacent}$$
$$\text{and} \quad N[v_i] \subseteq N[v_j] \quad \text{if } v_i \text{ and } v_j \text{ are adjacent.}$$

A graph is a threshold graph if and only if it has a threshold order (Mahadev & Peled 1995). This implies that the vertices of a threshold graph can be partitioned into a clique (the higher degree vertices) and an independent set (the lower degree vertices). Note however, that this partition is not exactly unique.

Theorem 3. *A graph* G $= (V, E)$ *is a special 2-threshold graph if and only if it has no induced* $2K_2$, C_5, *net, house, gem, octahedron, or a 4-wheel with a pendant vertex adjacent to the center, or a diamond with a pendant vertex attached to each vertex of degree three.*
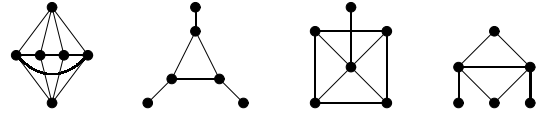

Figure 3: A octahedron, a net, a 4-wheel with a pendant adjacent to the center, and a diamond with a pendant vertex adjacent to each vertex of degree three

Proof. It is easy to check that the listed subgraphs are in the obstruction set. We prove the sufficiency. If a connected graph has no P_5 nor C_5 then it has a dominating clique (Bacsó & Tuza 1991, Brandstädt et al. 1999). That is, it has a clique C such that every vertex outside C has at least one neighbor in C. Since the graph has no $2K_2$ it has also no P_5 and, since there is also no C_5 there is a dominating clique.

Since there is no $2K_2$ there is at most one component with more than one vertex. If there is a component with more than one vertex then assume that there is a decomposition tree for that component. Otherwise start with a decomposition tree that consists of a single leaf and map any of the isolated vertices to that leaf. Color the vertex black. We add the isolated vertices one by one as black vertices as follows. Add a new root to the decomposition tree with a $\oplus$-operator. Add the isolated vertex as a right child and the original root as a left child.

Henceforth assume that G is connected.

Let C be a dominating clique of maximal cardinality. Among the dominating cliques of maximal cardinality choose C such that the number of edges with one endpoint in C and the other endpoint in $V - C$ is maximal.

We claim that $V = C$ or that $G - C$ is bipartite. For that, it is sufficient to show that there is no triangle. Assume that there is a triangle $\{x, y, z\}$ in $V - C$. Let X, Y and Z be the vertices in C that are adjacent to x, y and z respectively, but that are not adjacent to any other vertex of the triangle $\{x, y, z\}$. Let

$$X_2 = (N(y) \cap N(z) \cap C) \setminus N(x)$$

and define Y_2 and Z_2 similarly. Let N be the set of vertices in C adjacent to all three of x, y and z. Since there is no gem, there are no edges between X and Y_2, nor between X and Z_2, nor between Y and X_2, nor between Y and Z_2, nor between Z and X_2, nor between Z and Y_2. Since C is a clique at least one of every pair of these is empty. Since there is no house, there are no edges between X and Y, nor between X and Z, nor between Y and Z. Thus at least one of every pair is empty. Assume that $Y = Z = \varnothing$. If $X \neq \varnothing$ then $Y_2 = Z_2 = \varnothing$.

There is at most one vertex in C which is not adjacent to x or y or z, otherwise there is a $2K_2$. Assume that there is such a vertex p. Then $X = \varnothing$, otherwise there is a $2K_2$; let $\epsilon \in X$ then the edges (ϵ, p) and (y, z) form a $2K_2$. At least one of X_2, Y_2 and Z_2 is empty otherwise there is an octahedron. Say that $X_2 = \varnothing$. Obviously, $C \neq \{p\}$ since C is a dominating clique and p is not adjacent to x, y or z. Assume that there is a vertex $p' \notin C$ that is adjacent to p but not adjacent to x. Then p' is adjacent to y and to z otherwise there is a $2K_2$. If p' is adjacent to some vertex of Y_2 and to some vertex of Z_2 then there is an octahedron. Assume that p' is not adjacent to any vertex of Y_2. If $Y_2 \neq \varnothing$, then there is a gem in $Y_2 + \{z, x, y, p'\}$. Thus $Y_2 = \varnothing$. If there is a vertex in Z_2 that is adjacent to p' then there is a gem contained in $Z_2 + \{p', z, y, p\}$. So p' is nonadjacent to all vertices of Z_2. If $Z_2 \neq \varnothing$, then there is a gem induced by $Z_2 + \{y, p', z, x\}$. Thus

$Z_2 = \varnothing$. It follows that all vertices of $C - p$ are adjacent to x, to y and to z. If some vertex of N is adjacent to p' then there is a gem contained in $N + \{p, p', y, x\}$. Thus p' is nonadjacent to all vertices of N. Now there is a house induced by $N + \{y, p', p, x\}$, which is a contradiction.

Now assume that all neighbors of p are also adjacent to x. We can replace p in C by x. This gives a dominating clique C' of the same cardinality as C. The number of edges with one endpoint in C and the other endpoint in $V - C$ increases since x is furthermore adjacent to y and z.

Assume that every vertex in C is adjacent to at least one vertex of x, y and z. Assume $X \neq \varnothing$. Then it contains only one vertex $X = \{p\}$ otherwise there is a $2K_2$. Furthermore, $Y_2 = Z_2 = \varnothing$. Consider a neighbor $p' \neq x$ of p. If p' is not adjacent to y and not adjacent to z there is a $2K_2$ namely $\{(y, z), (p, p')\}$. If p' is adjacent to y and not adjacent to z there is a house or a gem induced by $\{z, p, p', x, y\}$. Thus every neighbor of p is a neighbor of both y and z. Notice that $N + X_2 \neq \varnothing$, otherwise y and z have no neighbor in C. If we replace p in C by $\{y, z\}$ we obtain a dominating clique of larger cardinality which is a contradiction.

Assume that $X = \varnothing$. Since there is no octahedron at least one of X_2, Y_2 and Z_2 is empty. Without loss of generality assume that $X_2 = \varnothing$. Then $C + \{x\}$ is also a (dominating) clique which contradicts the maximal cardinality of X.

This proves the claim that either $V - C$ is empty or that $G[V - C]$ is bipartite.

Since there is no $2K_2$ the bipartite graph is a difference graph (Hammer et al. 1990). Every induced subgraph without isolated vertices has a vertex in each side of the bipartition that is adjacent to all the vertices in the other side of the bipartition.

Assume that there is a component with more than one vertex. There is a unique partition of the vertices into two independent sets. Call these sets B (black) and W (white). We prove that $G[B + C]$ and $G[W + C]$ are threshold graphs. Assume that there are two black vertices x and y that have private neighbors x' and y' in C. That is, x' is a neighbor of x but not of y and y' is a neighbor of y but not of x. Since x and y are in a component of $G[B+W]$ there is an alternating path $P = [x, x_1, \ldots, x_k, y]$ of black and white vertices. Assume $k = 1$. If x_1 is not adjacent to x' and not adjacent to y' there is a C_5 which is a contradiction. If x_1 is adjacent to exactly one of x' and y' then there is a house. If x_1 is adjacent to both x' and y' then there is a gem.

We proceed by induction. Assume that there is a black vertex x_i in P that is adjacent to exactly one of x' and y'. Then we obtain a contradiction by considering one of the two subpaths $[x, x_1, \ldots, x_i]$ and $[x_i, \ldots, x_k, y]$. Assume that every black vertex in P, except x and y, is adjacent to both or to neither of x' and y'. Assume that there exist two white vertices x_i and x_j in P such that x_i is adjacent to x' but not to y' and x_j is adjacent to y' but not to x'. Since x_i and x_j are both white they are not adjacent. Then we obtain a contradiction by considering the subpath of P with terminals x_i and x_j. So we can assume that all the white vertices in P are adjacent to both or neither of x' and y' or, that they all have the same neighbor in $\{x', y'\}$. Assume that all the white vertices are adjacent to x' but not to y'. If the black vertex x_2 is not adjacent to x' then $\{x', x, x_1, x_2, x_3\}$ induces a house. If x_2 is adjacent to x' then this set induces a gem. The case where all the white vertices are adjacent to y' but not to x' is similar. Assume that all the white vertices are adjacent to both or neither of

x' and y'. Every edge of P must have at least one endpoint adjacent to one of x' or y' otherwise there is a $2K_2$. Assume that all the white vertices are adjacent to both x' and y'. Then $\{x', x, x_1, x_2, x_3\}$ induces a gem or a house. Let x_i be the first white vertex in P that is not adjacent to x' nor to y'. Assume $i = 1$. Then $\{x', y', x, x_1, x_2\}$ induces a house or there is a C_5 or a $2K_2$. The case where $i = k$ is similar. Assume that $1 < i < k$. Then x_{i-2}, x_{i-1} and x_{i+1} are adjacent to x'. Now $\{x', x_{i-2}, x_{i-1}, x_i, x_{i+1}\}$ induces a house.

This proves the claim that $G[B + C]$ and $G[W + C]$ are threshold graphs.

We prove that the neighborhoods of the black vertices and the neighborhoods of the white vertices are ordered by inclusion. Assume that there exist two black vertices x and y with private neighbors $x' \in C$ and $z \in W$. Since there is no $2K_2$ the vertex z is adjacent to x'. Because C is a dominating clique the vertex y has a neighbor $y' \in C$. The vertex y is not adjacent to x' and this implies that $y' \neq x'$. Since the neighborhoods in C of the black vertices are ordered by inclusion and since y is not adjacent to x', x is adjacent to y'. If z is adjacent to y' then $\{y', x, x', z, y\}$ induces a gem. If z is not adjacent to y' then this set induces a house. Similarly it holds true that no two white vertices have private neighbors.

This proves the claim that the neighborhoods of the black vertices and the neighborhoods of the white vertices are ordered by inclusion.

Consider a vertex $x \in C$ that has white neighbors and black neighbors. We prove that every white neighbor is adjacent to every black neighbor. Let α be a black neighbor and let β be a white neighbor and assume that α and β are not adjacent. Since α and β are in a component of $G - C$ there is an alternating path $P = [\alpha, x_1, \ldots, x_k, \beta]$ of black and white vertices. Since the neighborhoods of the black vertices are ordered by inclusion there is at most one black vertex x_i on this path $i.e.$, $i = 2$. Similarly, there is at most one white vertex x_j on this path, $i.e.$, $j = 1$. The black vertex α is not adjacent to β and the black vertex x_2 is adjacent to β and either $N(\alpha) \subseteq N(x_2)$ or $N(x_2) \subseteq N(\alpha)$. This implies that $N(\alpha) \subset N(x_2)$. Thus x_2 is adjacent to x. Similarly x_1 is adjacent to x. Thus $P + x$ induces a gem, which is a contradiction.

Let x and y be two vertices in C that have neighbors in B. We prove that

if $\varnothing \neq N(y) \cap B \subset N(x) \cap B$ then $N(x) \cap W \subseteq N(y) \cap W$.

Assume that x has a neighbor p in B that is not adjacent to y. Let q be a neighbor of y in B. Let r be a neighbor of x in W that is not adjacent to y. Since $G[B + C]$ is a threshold graph and since p is not adjacent to y the vertex q is adjacent to x. By the previous observation the vertices p and q are both adjacent to r. This implies that $\{x, y, q, r, p\}$ induces a gem.

Let $C^* \subseteq C$ be the set of vertices in C that have no neighbors in B or in W. We prove that $C^* = \varnothing$. Assume that $C^* \neq \varnothing$. Then $|C^*| = 1$ otherwise there is a $2K_2$ since there is an edge in $G[B + W]$. Let $C^* = \{c\}$. Let p be a black vertex with a minimal neighborhood and let q be a white vertex with a minimal neighborhood. Assume that p and q are not adjacent. Let $p' \in C$ be a neighbor of p and let $q' \in C$ be a neighbor of q. Then $p' \neq q'$ otherwise p and q are adjacent. Let a be a black vertex with a maximal neighborhood and let b be a white vertex with a maximal neighborhood. Then a and b are adjacent. Assume that $a \neq p$ and $b \neq q$. The vertex a is adjacent to p' and the vertex b is adjacent to q'. If a is not adjacent to q' and b is not adjacent to p' then $\{a, b, q', p', c\}$ induces a house. If a is adjacent to q' then a is adjacent to q and $\{q', q, a, p', c\}$ induces a gem. Assume that $a = p$, $i.e.$, assume that there is

only one black vertex. Then p is adjacent to q because $G[B + W]$ is connected. This contradicts the assumption that p and q are not adjacent. Assume that p and q are adjacent. Then $G[B + W]$ is complete bipartite. Assume that $p' \neq q'$. If p is not adjacent to q' and q is not adjacent to p' then $\{c, p, q, q', p'\}$ induces a house. If p is adjacent to q' and q is not adjacent to p' then $\{q', q, p, p', c\}$ induces a gem. Assume that p is adjacent to q' and that q is adjacent to p'. Then all the black vertices and all the white vertices are adjacent to p' and to q'. Assume that there is a black vertex $a \neq p$ and a white vertex $b \neq q$. Then $\{a, b, p, q, p', c\}$ induces a 4-wheel with a pendant. Assume that there is only one black vertex p. Assume that there is a vertex $\delta \in C - C^*$ which is not adjacent to p. Then δ is adjacent to a white vertex b and $\{p', c, \delta, b, p\}$ induces a gem. Assume that there is an isolated vertex x in $G - C$ adjacent to c. Then $\{(p, q), (c, x)\}$ is a $2K_2$. Thus $N(c) \subset N(p)$. Replace c by p. Then we obtain a dominating clique C' with the same cardinality as C but there are more edges with one endpoint in C' and the other endpoint in $V - C'$ since p is furthermore adjacent to q. This contradicts the assumption that C maximizes the number of edges with one endpoint in C and the other in $V - C$. Assume $p' = q'$. Then $G[B + W]$ is complete bipartite. If there are black and white vertices $a \neq p$ and $b \neq q$ then $\{a, b, p, q, p', c\}$ induces a 4-wheel with a pendant. Assume that there is only one black vertex p. If there is a vertex $\delta \in C - C^*$ that is not adjacent to p then δ is adjacent to a white vertex b. Then $\{p', c, \delta, b, p\}$ induces a gem. There is no isolated vertex in $G - C$ adjacent to c otherwise there is a $2K_2$. As above we can replace c by p and obtain a dominating clique C' of the same cardinality as C but with more edges than C with one endpoint in C and the other endpoint in $V - C'$.

This proves the claim that every vertex of C has a neighbor in $B + W$.

Define an ordering on the vertices of $C + B + W$ as follows. Let $x \preceq y$ if

1. $x \in B$ and $y \in B$ and $N(y) \subseteq N(x)$,

2. $x \in B$ and $y \in C$ and $y \in N(x)$,

3. $x \in B$ and $y \in W$ and $y \in N(x)$,

4. $x \in C$ and $y \in C$ and $N(x) \cap B \subseteq N(y) \cap B$ and $N(y) \cap W \subseteq N(x) \cap W$,

5. $x \in C$ and $y \in W$ and $y \in N(x)$,

6. $x \in W$ and $y \in W$ and $N(x) \subseteq N(y)$,

and the inverse conditions for the remaining pairs. For every pair x and y in $C + B + W$ either $x \preceq y$ or $y \preceq x$. We proved that the relation is transitive. It is not necessarily antisymmetric. Define an equivalence relation on the vertices as follows. Two vertices of $C + B + W$ are equivalent if they are in the same set of the partition and, if they have the same open or closed neighborhood. Then the relation $\preceq$ induces a linear order on the equivalence classes. Notice that we could define the reversed order likewise.

Let I be the subset of vertices in C that have no black neighbors. Let J be the subset of vertices in C that have no white neighbors. The vertices of I appear before all the black vertices in the linear order and the vertices of J appear after all the white vertices in the linear order. Thus $I \cap J = \varnothing$ since $G[B + W]$ is connected. Also $I \neq \varnothing$ and $J \neq \varnothing$ since C is a maximal clique.

Let x be an isolated vertex in $G - C$. We prove that $N(x)$ is contained in I or it is contained in J. Let p be a black vertex with a minimal neighborhood and let q be a white vertex with a minimal neighborhood. Assume that p and q are not adjacent. Let $p' \in C$ be a neighbor of p and let $q' \in C$ be a neighbor of q. Then $p' \neq q'$ and p is not adjacent to q' and q is not adjacent to p'. Assume that there is an isolated vertex x in $G - C$ adjacent to p' and to q'. Let a be a black vertex with a maximal neighborhood and let b be a white vertex with a maximal neighborhood. Then a and b are adjacent. The vertex a is adjacent to p' and the vertex b is adjacent to q'. If a is not adjacent to q' and b is not adjacent to p' then there is a house $\{a, b, q', p', x\}$. If a is adjacent to q' then a is adjacent to q and there is a gem $\{q', q, a, p', x\}$. If there is only one black vertex p then p is adjacent to q since $G[B + W]$ is connected. This contradicts the assumption that p and q are not adjacent. Assume that p and q are adjacent. Then $G[B + W]$ is complete bipartite. Assume that $p' \neq q'$. If p is not adjacent to q' and q is not adjacent to p' then $\{p, q, p', q', x\}$ induces a house. If p is adjacent to q' but q is not adjacent to p' then $\{q', q, p, p', x\}$ induces a gem. Assume that p is adjacent to q' and that q is adjacent to p'. There is a vertex $\delta_1 \in C$ that is adjacent to p but not to q and there is a vertex $\delta_2 \in C$ that is adjacent to q but not to p. If δ_1 is adjacent to x then there is a gem $\{p', x, \delta_1, p, q\}$. If δ_2 is adjacent to x then there is a gem $\{q', x, \delta_2, q, p\}$. If δ_1 is not adjacent to x and δ_2 is not adjacent to x then there is a 4-wheel with a pendant attached to its center $\{p, \delta_1, \delta_2, q, p', x\}$. The same contradiction follows when $p' = q'$.

Assume that x is not adjacent to any neighbor of p and that x is not adjacent to any neighbor of q. Assume that p and q are not adjacent. Let $p' \in C$ be a neighbor of p and let $q' \in C$ be a neighbor of q. Let $\delta \in C$ be a neighbor of x. Then $\{p, q, x, p', q', \delta\}$ is a net. Assume that p and q are adjacent. Let $\delta_1 \in C$ be a neighbor of p that is not a neighbor of q and let $\delta_2 \in C$ be a neighbor of q that is not a neighbor of p. Let $\delta \in C$ be a neighbor of x. Then $\{\delta, \delta_1, \delta_2, q, p\}$ induces a house. This proves that x has common neighbors in C with exactly one of p and q. Assume that x has no common neighbors with q. We proceed by induction on the number of white vertices. Remove q. By induction it follows that x has no common neighbors with any white vertex.

This proves the claim that $N(x)$ is contained in I or in J for any isolated vertex x of $G - C$.

We prove that the neighborhoods of isolated vertices of $G - C$ that have neighbors in I are ordered by set inclusion. Consider three isolated vertices x, y and z of $G - C$ that have neighbors in I. Assume that x and y have private neighbors x' and y' in C. Assume that z is adjacent to x' and to y'. There exists a vertex $\delta \in C$ that is not adjacent to z. If x is adjacent to δ there is a gem $\{x', x, \delta, y', z\}$. If y is adjacent to δ there is a gem $\{y', y, \delta, x', z\}$. If δ is not adjacent to x and not to y there is a diamond with two pendant vertices $\{z, x', y', \delta, x, y\}$. Assume that z is not adjacent to x' and that z is not adjacent to y'. Assume that z has a neighbor z' that is not adjacent to x and that is not adjacent to y. Then $\{x', y', z', x, y, z\}$ is a net. If z has a neighbor z_1 that is adjacent to x but not to y and a neighbor z_2 that is adjacent to y but not to x then there is a gem $\{z_1, x, x', z_2, z\}$. Assume that z is adjacent to x' but not adjacent to y'. Assume that x has a neighbor x'' that is not adjacent to z and that z has a neighbor z' that is not adjacent to x. Then $\{x', z, z', x'', x\}$ is a gem. Thus the neighborhood of z is comparable with the neighborhood of exactly one of x and y and it is disjoint with the neighborhood of the other one.

This proves the claim that the neighborhoods of the isolated vertices with neighbors in I are ordered by set inclusion. A similar proof shows that the neigh-

borhoods of the white vertices and of the isolated vertices with their neighborhoods in I and, the neighborhoods of the black vertices and of the isolated vertices with their neighborhoods in J are ordered by set inclusion.

Color the vertices of C white and equip them with a $\otimes_w$-operator. If there is a component in $G - C$ with more than one vertex then color the vertices of B black and equip them with a $\otimes_w$-operator. Color the vertices of W white and equip them with a $\oplus$-operator. Color the isolated vertices of $G - C$ that have their neighbors in I white and equip them with a $\oplus$-operator. Color the isolated vertices of $G - C$ that have their neighbors in J black and equip them with a $\otimes_w$-operator. If $G - C$ has no component with more than one vertex then let I and J be maximal disjoint neighborhoods in C. We proved that there is a linear ordering of the vertices that satisfies the neighborhood conditions.

This proves the theorem. $\qquad\square$

Remark 1. *A class of graphs that is closely related to the special 2-threshold graphs is the class of probe threshold graphs (Bayer et al. 2009, Chandler et al. 2009). The obstruction set for probe threshold graphs appears in (Bayer et al. 2009) (without proof). That the classes are different is illustrated by the complement of* $2P_3$. *This graph is not a probe threshold graph but it is a special 2-threshold graph. The gem is an example of a probe threshold graphs that is not a special 2-threshold graph.*

Lemma 1. *The special 2-threshold graphs have rankwidth one. They can be recognized in linear time.*

Proof. A graph G is distance hereditary if in every induced subgraph G[W] for every pair of vertices in a component of G[W] the distance between them is the same as in G (Howorka 1977). The distance-hereditary graphs can be characterized as follows. Every induced subgraph has an isolated vertex, or a pendant vertex, or two vertices x and y such that every other vertex z is adjacent to both x and y or to neither of them. A pair of vertices like that is called a twin. It is easy to check that the special graphs satisfy this property. The distance-hereditary graphs are exactly the graphs of rankwidth one (Oum 2005). A rank-decomposition tree can be found in linear time (Hammer & Maffray 1990). We can formulate the existence of an induced subgraph that is isomorphic to one of the graphs in the obstruction set in monadic second-order logic without quantification over edge-sets. This proves the lemma. $\qquad\square$

4 Partitioned black-and-white threshold graphs

In this section we turn our attention to the partitioned 2-threshold graphs, that is, we assume that the coloring is a part of the input.

Lemma 2. *If G is 2-threshold then the following statements hold true.*

(a) *G has at most two components with at least two vertices.*

(b) *If G has two components with at least two vertices, then one is a threshold graph and the other is a special 2-threshold graph.*

Proof. Assume that G has more than two components with at least two vertices. Then G has an induced $3K_2$. It is easy to check that $3K_2$ is not a 2-threshold graph.
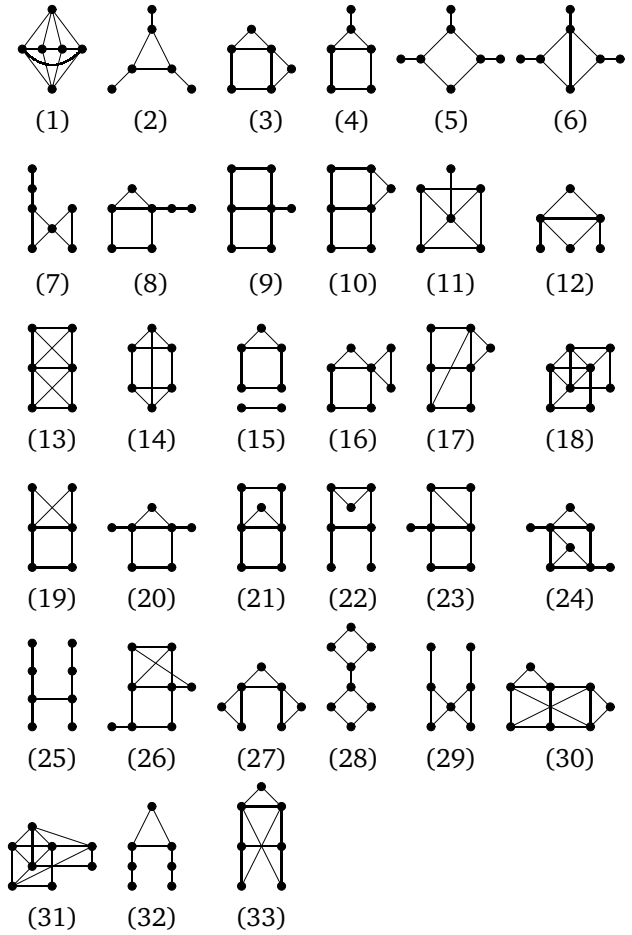

Figure 4: The set of forbidden induced subgraphs of 2-threshold graphs contains the gem, $3K_2$, C_5, C_6, C_7, P_7, $2C_4$, $2P_4$, $C_4 + P_4$ and the graphs listed above.

Assume that there are two components C_1 and C_2 with at least two vertices. Let x and y be adjacent vertices in C_1. First assume that x is black and y is white. Assume that C_2 has a black vertex and a white vertex. Then it has an edge with one black endpoint and one white endpoint. We find a $2K_2$ and both edges have one black endpoint and one white endpoint. It is easy to check that there is no elimination ordering for a $2K_2$ that is colored like that. Thus C_2 is monochromatic which implies that C_2 induces a threshold graph.

Now assume that x and y are both black. Consider a decomposition tree for $C_2 + \{x, y\}$. Assume that x is closer to the root than y. Then the operator of x is a $\otimes_b$-operator since x is adjacent to y. All the operators that appear closer to the root than the operator of y are either $\oplus$ or $\otimes_w$ otherwise they are adjacent to y. All the vertices further from the root than y are white, otherwise they are adjacent to x. So we may replace any $\otimes_b$-operator that is further from the root than the operator of y by a $\oplus$-operator. Thus the tree for C_2 now only has $\oplus$- and $\otimes_w$-operators; *i.e.*, the graph induced by C_2 is a special threshold graph. $\quad\square$

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The union of G_1 and G_2 is the graph

$$G_1 + G_2 = (V_1 + V_2, E_1 + E_2).$$

The join of G_1 and G_2 is the graph obtained from the union by adding an edge between every vertex of V_1 and every vertex of V_2. We denote the join by $G_1 \times G_2$.

Lemma 3. *Let* G *be a 2-threshold graph. The graph induced by every nonempty neighborhood is one of the following.*

(a) *a threshold graph, or*

(b) *the union of two threshold graphs, or*

(c) *the join of two threshold graphs.*

Proof. Consider a black-and-white coloring of the vertices and a decomposition tree for G. Let x and y be two vertices. We write $x \prec y$ if x is closer to the root than y in the decomposition tree. We write $o(x)$ for the operator in the tree that is adjacent to x.

Let x be a vertex and assume that x is black. We consider three cases.

Case 1: $o(x) = \otimes_b$. In this case $N(x) = T_1 + T_2$ where T_1 and T_2 are defined as follows.

$$T_1 = \{\, y \in V \mid y \prec x \text{ and } o(y) = \otimes_b \,\}$$
$$T_2 = \{\, y \in V \mid x \prec y \text{ and } y \text{ is black} \,\}.$$

Notice that every vertex of T_1 is adjacent to every vertex of T_2. All vertices of T_2 are black thus T_2 induces a threshold graph. The black vertices of T_1 are a clique and the white vertices are an independent set. Every white vertex is adjacent to those black vertices that are further from the root. This implies that T_1 is a threshold graph (if it is nonempty) (Mahadev & Peled 1995).

Case 2: $o(x) = \oplus$. In this case the neighbors of x are the vertices of T_1 described above.

Case 3: $o(x) = \otimes_w$. Now $N(x) = T_1 + T_2$ where T_1 and T_2 are defined as follows.

$$T_1 = \{\, y \in V \mid y \prec x \text{ and } o(y) = \otimes_b \,\}$$
$$T_2 = \{\, y \in V \mid x \prec y \text{ and } y \text{ is white} \,\}.$$

In this case there are no edge between vertices of T_1 and vertices of T_2. If $T_2 \neq \varnothing$ then the graph induced by T_2 is a threshold graph since all vertices are white. We proved above that T_1 is also a threshold graph.

A similar analysis holds when x is a white vertex. This proves the lemma. □

The complement of a graph $G = (V, E)$ is the graph that has V as its vertices and that has those pairs of vertices adjacent that are not adjacent in G. We denote the complement of G by $\bar{G}$.

Let G be a graph. A vertex of G is good if its neighborhood is empty or, a threshold graph or, the union of two threshold graphs or the join of two threshold graphs. The graph G is good if every vertex of G is good. By Lemma 3 a 2-threshold graph is good. Consider a vertex x in G. A local complementation at x is the operation which complements the graph induced by $N(x)$. Note that the class of good graphs is hereditary and closed under local complementations.

Theorem 4. *A graph* G *is good if and only if* G *has no induced gem or,* $C_4 + K_1$ *with a universal vertex or, the local complement of this with respect to the universal vertex or,* $3K_2$ *with a universal vertex or, an octahedron with a universal vertex.*

Proof. It is easy to check that the universal vertex of each of the mentioned graphs is not good. We prove the converse. Let x be a vertex with a nonempty neighborhood. Since there is no gem the graph induced by $N(x)$ is a cograph. Assume that this graph is disconnected. Let $C_1, \ldots, C_t$ be the components of
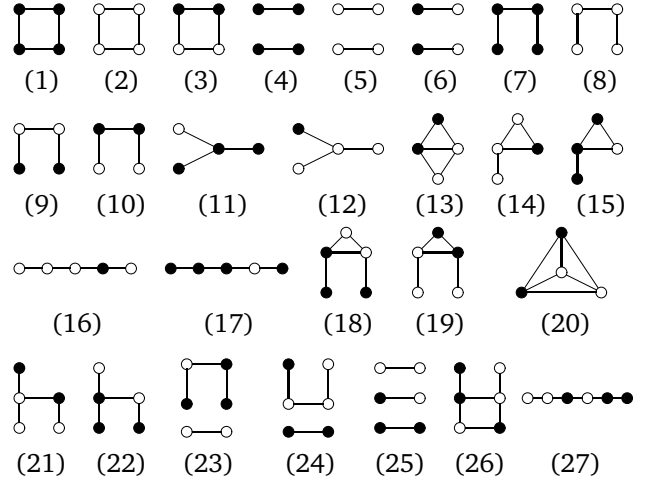

Figure 5: Forbidden induced subgraphs of partitioned black-and-white threshold graphs

the graph induced by $N(x)$. There is no $3K_2$ in the graph induced by $N(x)$ so there are at most two components with more than one vertex. Since there is no 4-wheel with a pendant vertex attached to its center the graph induced by each component C_i has no C_4. Assume that C_i has an induced $2K_2$. Since the graph induced by C_i is connected and since there is no P_4 in $G[C_i]$ this graph has a butterfly. Then G has an induced subgraph that is isomorphic a butterfly with a universal vertex. The butterfly is the complement of $C_4 + K_1$.

Now assume that the graph induced by $N(x)$ is connected. Consider the graph obtained by complementing $N(x)$. By the argument above x is good in this graph. This implies that x is also good in G. □

Theorem 5. *A partitioned graph is a 2-threshold graph if and only if it does not contain any of the graphs of Figure 5 as an induced subgraph.*

Proof. It is easy to see that none of the graphs in Figure 5 is a 2-threshold graph. We prove the converse. Let G be a partitioned graph with none of the graphs of Figure 5 as an induced subgraph. It can be checked that none of the graphs in Figure 4 is an induced subgraph of the underlying unpartitioned graph.

The case where there are two components with at least two vertices is easy to check. Assume that G is connected. We prove that there exists a vertex x such that $N(x)$ or $N[x]$ is exactly the set of all the white vertices or all the black vertices in G. Assume that G has a house $H = [1; 2, 3, 4, 5]$ where $[2, 3, 4, 5]$ is the induced 4-cycle in H and vertex 1 is the rooftop of H adjacent to vertices 2 and 5. Assume that vertices 1, 2, 3 and 5 are colored black and that vertex 4 is colored white. Let $S \subseteq V - H$ be the set of vertices that are adjacent to 1, 2, 3 and 5, and nonadjacent to 4. Add vertex 2 also to S. We claim that S contains a vertex x such that $N(x)$ or $N[x]$ is the set of all black vertices in G. It can be checked that every black vertex is in S or is adjacent to a vertex in S. The graph G[S] is a threshold graph since G has no octahedron, or gem or a $K_2 \times 2K_2$. The black vertices in S are a clique and the white vertices are an independent set. It is now easy to check that there exists a vertex $x \in S$ such that $N(x)$ or $N[x]$ is the set of black vertices in G.

The other colorings of the house can be dealt with in a similar fashion.

Assume that G contains no house. Assume that there is a $2K_2$; $(p, q) + (r, s)$. Assume that p and q are black and that r and s are white. Consider a shortest

path between $\{p, q\}$ and $\{r, s\}$. Assume that this is a P_6;
$$P = [p, q, a, b, r, s].$$
Then a is black and b is white. Assume that q has a white neighbor α and that r has a black neighbor β. Then
$$N(\alpha) \cap P = \{p, q, a\} \quad \text{and} \quad N(\beta) \cap P = \{b, r, s\}.$$
Furthermore, α and β are adjacent. Then $\{q, a, b, \alpha, \beta\}$ induces a house.

Assume that q has only black neighbors. It is easy to check that there is a vertex $x \in N[q]$ such that $N(x)$ or $N[x]$ is the set of black vertices.

Other cases can be dealt with in a similar manner.

Assume that there is no house and no $2K_2$. Then G is special. It is easy to check that the claim holds true in that case. This proves the theorem. □

5 The switching class of threshold graphs

If we allow a $\otimes$-operator, which adds a universal vertex, then we obtain a slightly bigger class of graphs. We call these graphs extended 2-threshold graphs. It is easy to see that if G is an extended 2-threshold graph then so is its complement.

We obtain a class of graphs that is contained in the 2-threshold graphs if we allow only $\otimes_w$-operators and $\otimes_b$-operators. We call these graphs restricted 2-threshold graphs. It is easy to see that also the restricted 2-threshold graphs are self-complementary. We prove in this section that the restricted 2-threshold graphs are exactly the graphs that are switching equivalent to threshold graphs.

Definition 4. *Let* $G = (V, E)$ *be a graph and let* $S \subseteq V$. *Switching* G *with respect to* S *is the operation that adds all edges to* G *between nonadjacent pairs with one element in* S *and the other not in* S *and that removes all edges between pairs with one element in* S *and the other not in* S.

Switching is an equivalence relation on graphs. The equivalence classes are called switching classes. The work on switching classes was initiated by van Lint and Seidel (van Lint & Seidel 1966, Seidel 1991).

Theorem 6. *If* G *is in the switching class of a threshold graph then* G *is a restricted 2-threshold graph.*

Proof. Assume that G is obtained by switching a threshold graph H with respect to some set S of vertices. Consider the decomposition tree for the threshold graph H. Color the vertices of S white and color the other vertices black. Change the $\otimes$- and $\oplus$-operators in the tree as follows.

1. If a vertex x is black and $o(x) = \otimes$ then change $o(x)$ to $\otimes_b$.

2. If a vertex x is black and $o(x) = \oplus$ then change $o(x)$ to $\otimes_w$.

3. If a vertex x is white and $o(x) = \otimes$ then change $o(x)$ to $\otimes_w$.

4. If a vertex x is white and $o(x) = \oplus$ then change $o(x)$ to $\otimes_b$.

It is easy to see by induction on the height of the decomposition tree that these operations change the decomposition tree into a decomposition tree for G. □
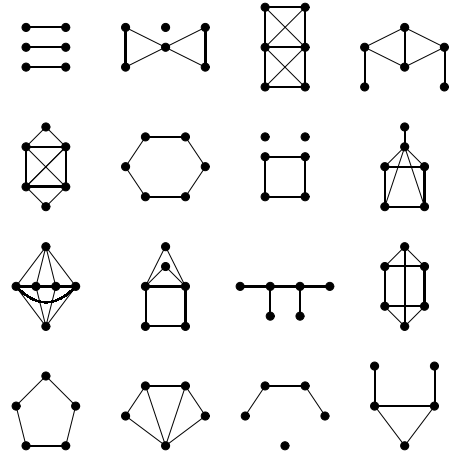

Figure 6: Forbidden subgraphs of the switching class of threshold graphs. Each of them is switching equivalent to $3K_2$, C_5 or $C_4 + 2K_1$.

Cameron examined the switching class of cographs (Cameron 1994). A graph is a cograph if every induced subgraph with at least two vertices is either disconnected or its complement is disconnected (Corneil et al. 1981). Thus the class of cographs contains the threshold graphs. The switching class of cographs are characterized as the graphs without induced C_5, bull, gem, or co-gem (Hung & Kloks 2010). In other words, a graph is switching equivalent to a cograph if and only if it has no induced subgraph that is switching equivalent to C_5. Thus these graphs are also forbidden for the graphs that are switching equivalent to a threshold graph.

Theorem 7. *A graph* $G = (V, E)$ *is switching equivalent to a threshold graph if and only if* G *has no induced subgraph which is switching equivalent to* $3K_2$, C_5 *or* $C_4 + 2K_1$.

Proof. It is easy to check the necessity. We prove the sufficiency. Since the graph has no induced subgraph which is switching equivalent to C_5 the graph G is switching equivalent to a cograph H. If H is connected then we can switch it to a disconnected cograph by switching it with respect to one of the components of the complement. Henceforth we assume that H is disconnected. Let $C_1, C_2, \ldots$ be the components of H. There are at most two components with more than one vertex, otherwise the graph has an induced $3K_2$. Furthermore, there can be at most one component which is not a threshold graph already. First assume that every component is a threshold graph. Then switch the graph with respect to a maximal clique in one of them. It is easy to check that this produces a threshold graph. Assume that C_1 does not induce a threshold graph.

By induction we can assume that there is a subset S of C_1 such that switching $H[C_1]$ with respect to S produces a threshold graph T. The threshold graph T has a universal or isolated vertex x. We may assume that $x \notin S$, otherwise we can switch $H[C_1]$ with respect to $C_1 - S$. If x is isolated in T then $S = N_H(x)$ and if x is universal in T then $S = C_1 - N_H[x]$.

Assume that $S = N_H(x)$. If $C_1 = N_H[x]$ then $H[C_1]$ is a threshold graph. Assume that $C_1 - N_H[x] \neq \varnothing$. Let $\Delta = C_1 - N_H[x]$. For every pair of vertices a and b in Δ their neighborhoods in S are ordered by inclusion, otherwise the switch produces a C_4, or P_4 or $2K_2$. Since $H[C_1]$ is connected and has no induced P_4 every vertex of Δ has a neighbor in S. Let $S' \subseteq S$ be the subset of vertices in S that have a neighbor in Δ.

Then every pair with one element in S' and one element in $S - S'$ is adjacent, since there is no P_4 in $H[C_1]$. The switch makes every vertex of $S - S'$ adjacent to every vertex of Δ. Since the C_4 is forbidden at least one of $S - S'$ and $S' + \Delta$ is a clique.

Assume that there is an edge (p, q) in Δ and assume that p has a neighbor p' in S that is not a neighbor of q. Then $[q, p, p', x]$ is a P_4 in H which is forbidden. Thus all vertices of every component of Δ have the same neighbors in S. Call the sets of vertices of S' that have the same neighbors in Δ the classes of S'. Call the unions of those components of Δ that have the same neighbors in S' the classes of Δ.

Assume that there are two classes P and Q in S' and assume that P is adjacent to a class P' of Δ which is not adjacent to Q. Let Q be adjacent to a class Q' of Δ. Then P is also adjacent to Q'. Since $[P', P, Q', Q]$ has no induced P_4, every vertex of P is adjacent to every vertex of Q. Furthermore, since there is no C_4 in the switched graph, at least one of the classes P and Q is a clique. Thus all of S' except possibly one class is a clique.

Since there is no $2K_2$ in the switched graph at most one of the components of Δ has more than one vertex. Furthermore, if there is a component with more than one vertex then it has the least number of neighbors in S' among all components of Δ, otherwise the switch produces a $2K_2$.

Every pair of nonadjacent classes in S' and Δ become adjacent in the switched graph. Thus at least one of them is a clique. If there is a class in S' that is not a clique then it is adjacent to all except possibly one class of Δ otherwise, there is a C_4 in the switched graph. Consider a component P in Δ that has more than one vertex and a class Q in S' that is not a clique. If P and Q are not adjacent then they are adjacent in the switched graph, thus P is a clique. If P and Q are adjacent then Q is an independent set, otherwise the switched graph has a $2K_2$.

If $S - S'$ is not a clique, then $S' + \Delta$ is a clique in the switched graph. Thus S' and Δ are disjoint cliques in $H_1[C_1]$. This contradicts that C_1 is a component of H and $H[C_1]$ is not a threshold graph. Thus $S - S'$ is empty or else it is a clique.

Assume that there is a class P in Δ which contains an edge. Let Q be the class in S' that is adjacent to P. Then Q is an independent set, otherwise there is a $2K_2$ in the switched graph. This class Q is adjacent to all vertices of Δ, since P has the least neighbors in S' among all classes in Δ. Thus Q is an independent, universal set of $H[C_1]$.

Assume that there is a component P in Δ that contains an edge. Let Q be the neighborhood of P in S'. Then $S - Q = \varnothing$ otherwise, H contains a graph which is switching equivalent to $3K_2$. Assume that $S - Q = \varnothing$ and that Q is an independent set. If Q contains more than one vertex then $V - C_1$ is a clique. Switch H with respect to

$$Q + (V - C_1).$$

This gives a threshold graph.

Assume that Δ is an independent set. Let Q be a class in S' which is not a clique. Then every vertex of Q is adjacent to all other vertices of C_1 except possibly one vertex in Δ. If there is a vertex in Δ which is not adjacent to Q then H contains a $C_4 + 2K_1$ which is a contradiction. Thus Q is adjacent to all vertices of $C_1 - Q$.

First assume that Q is an independent set. Switch H with respect to

$$Q + (V - C_1).$$

The set Q becomes a set of isolated vertices. The vertex x is adjacent to the clique $S - Q$. The graph induced by $\Delta + (S - Q)$ is a threshold graph. The set $V - C_1$ becomes a clique of vertices adjacent to all vertices in $C_1 - Q$. Thus H switches to a threshold graph.

Assume that Q is not an independent set. Switch H with respect to

$$Q + (V - C_1).$$

This gives two disjoint threshold graphs Q and $V - Q$. As shown at the start, we can switch the graph to a threshold graph.

The case where $S = C_1 - N[x]$ is similar. $\qquad\square$

Theorem 8. *A graph is switching equivalent to a threshold graph if and only if it is a restricted 2-threshold graph.*

Proof. It is easy to check that none of the forbidden induced subgraphs of the switching class of threshold graphs is restricted 2-threshold. $\qquad\square$

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